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Asking Converse Questions and Looking for Extensions to Gauss's Method for Summing Arithmetic Progressions

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Abstract: *Posing good problems is important for learning, teaching and research in mathematics. In this paper, the converse problem posing strategy is applied to Gauss's method that has been used to obtain the summation formula of an Arithmetic Progression. The work here serves as a simple but typical example to demonstrate the use of this strategy. The results obtained may also help the reader see to what extent Gauss's method can be applied, thus enriching one's understanding of this famous method.*

Introduction

Possibly the most well known story about the great German mathematician Carl Friedrich Gauss is that he astonished his teacher by instantly obtaining the sum of all numbers from 1 to 100 soon after the teacher gave the problem (see, for example, Borowski and Borwein, 1999). Gauss's method actually applies to an arbitrary Arithmetic Progression and was used to obtain the general formula of such sums.

Few people would doubt that being able to pose good problems is important for learning, teaching and research in mathematics. However, finding a quality problem is not easy. Besides a thorough understanding of the content materials, it also needs proper training and practice. A number of different problem posing methods have been investigated and suggested by several authors (see, for example, Brown and Walter (1990), Mason, Burton, and Stacey (1998)). One of the popularly used strategies is to formulate the *converse of a valid proposition*, and ask whether the converse statement is true. Roughly speaking, if a proposition is in the form 'If p then q', then the converse statement could be stated as: 'If q then p'.

However, very often a given proposition may not be formulated in the above standard form. In such a case, we have to modify it before we can pose its converse problem. In this paper, we shall apply the converse problem posing strategy to Gauss's method. We hope the work here will serve as a simple but typical example to demonstrate the use of this strategy. The results obtained here may also help the reader see to what extent Gauss's method can be applied, and thus deepen one's understanding of this famous method.

It was reported that Gauss obtained the summation of all numbers from 1 to 100 by adding 1 and 100 to get 101, 2 to 99 to obtain a second 101, and continuing this kind of addition until adding 55 to 51 to get the fiftieth 101. Since there are altogether fifty 101s, the summation is $50 \times 101 = 5050$. We can view the method in a slightly different way as follows. We reverse the sequence 1, 2, …, 100 to obtain another sequence 100, 99, …, 2, 1. Then we add up the corresponding terms of these two sequences to get a constant sequence 101, 101, …, 101, the summation of which is 100×101. Thus the summation of the original sequence is half of 100×101. From this second view of the method, we can obtain the summation of the first n terms of the sequence 1, 2, 3, …, k, …, for any n, and it is easy to see that the summation is $\frac{n}{2}(n+1)$.

In general, if a sequence $\{a_i\}_{i=1}^{\infty}$ satisfies the condition

$$a_1 + a_n = a_i + a_{n-i+1},\ \ 1 \le i \le n, \quad \text{for every positive integer } n,$$

then, as will be shown at the beginning of the next section, we can use Gauss's method to obtain the sum of its first n terms.

Every Arithmetic Progression satisfies the condition above. A natural question follows: Can Gauss' method be applied to non-Arithmetic Progressions? Or more specifically, if a sequence $\{a_i\}_{i=1}^{\infty}$ satisfies the above condition, must it be an Arithmetic Progression? In the first section of the paper we shall answer this question. In the next section we extend the scope to investigate the continuous counterpart of Gauss's method. In this case we consider the area of the region between $x = 0$, $x = a$, $y = 0$ and $y = f(x)$, where f is a function defined on the real line satisfying some designated conditions.

The final section some guidelines are provided to teachers on how the results in sections 1 and 2 can be explored and discovered in the classroom.[1]

The Discrete Case

Recall that a sequence of real numbers $\{a_i\}_{i=1}^{\infty}$ is an Arithmetic Progression (AP) if there is a number d such that $a_{i+1} = a_1 + id$ holds for each i = 1, 2, ….

[1] In these sections, theorems and corollaries are considered together and numbered consecutively, thus Theorem 1, Corollary 2, Theorem 3, and so on. Problems are numbered separately, while examples and exercises are not numbered at all. Note: The box symbol □ is used to indicate the end of a proof.

Example: The following two sequences are APs:

(i) 1, 2, 3, 4, …, n, …

(ii) 5, 7, 9, 11, …, $2k+1$, …, $k \geq 2$.

It is a simple exercise to prove by induction that if $\{a_i\}_{i=1}^{\infty}$ is an AP, then the following condition, which we shall label as (1), holds:

for every positive integer n, $a_1 + a_n = a_i + a_{n-i+1}$ holds for all i, $1 \leq i \leq n$ (1)

From here, it follows that

$$\sum_{i=1}^{n} a_i = \tfrac{1}{2}\left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i\right) = \tfrac{1}{2}\sum_{i=1}^{n}(a_i + a_{n-i+1}) = \tfrac{n}{2}(a_1 + a_n)\,.$$

In particular, if the sequence is an AP with common difference d, we may proceed as follows:

$$\sum_{i=1}^{n} a_i = \tfrac{n}{2}(a_1 + a_n) = \tfrac{n}{2}(a_1 + a_1 + (n-1)d) = \tfrac{n}{2}(2a_1 + (n-1)d)\,.$$

Of course, we may use the same formula $\sum_{i=1}^{n} a_i = \tfrac{n}{2}(a_1 + a_n)$ for finding the sum to n terms of any sequence $\{a_i\}_{i=1}^{\infty}$ satisfying condition (1). (We remark here that a key feature of (1) is that it holds for ALL positive integers n. Thus, the formula will apply irrespective of where we 'truncate' the sequence.) This leads to the first problem:

Problem 1: Suppose $\{a_i\}_{i=1}^{\infty}$ is a sequence satisfying (1). Must $\{a_i\}_{i=1}^{\infty}$ be an AP?

In fact, in order to see whether Gauss' method can be applied to a larger class of sequences, we shall consider the condition (2) below which is more general than (1). The condition (1) requires that $a_1 + a_n = a_i + a_{n-i+1}$ holds for every n and $i < n$, while condition (2) requires that $a_1 + a_n = a_i + a_{n-i+1}$ holds for every multiple n of a designated integer m and each $i \leq n$. Obviously, (2) is weaker than (1). In the following theorem, we prove that (2) is equivalent to the combination of two conditions, which implies that the sequence $\{a_i\}_{i=1}^{\infty}$ can be regrouped into a sequence of blocks of equal number of terms, so that the first block satisfies (1) and for each i, the subsequence obtained by taking the ith term from each block is an AP.

Theorem 1: Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Fix a positive integer m. Consider the following condition:

for every multiple n of m, $a_i + a_{n-i+1} = a_1 + a_n$ holds for all i, $1 \le i \le n$. (2)

Then (2) is satisfied by $\{a_i\}_{i=1}^{\infty}$ if and only if the following two conditions are satisfied:

(i) the subsequence $\{a_{i+rm}\}_{r=1}^{\infty}$ for each i, $1 \le i \le m$, is an AP with common difference $a_{m+1} - a_1$, and

(ii) $a_i + a_{m-i+1} = a_1 + a_m$ for all i, $1 \le i \le m$.

Proof [Sufficiency] Suppose the conditions (i) and (ii) are satisfied. We need to show that (2) holds. For a given i, the proof consists of 'shifting' the terms a_i and a_{n-i+1} to the first block by making use of conditions (i) and (ii). Let $n = km$ and $d = a_{m+1} - a_1$. Then, we have for each i, $1 \le i \le n$,

$$\begin{aligned} a_i + a_{n-i+1} &= a_i + a_{km-i+1} \\ &= a_{rm+i'} + a_{km-rm-i'+1}, \text{ where } i' = i - rm \text{ and } 1 \le i' \le m \\ &= a_{rm+i'} + a_{(k-r-1)m+(m-i'+1)} \\ &\qquad \text{(We may assume } i \le n - i + 1 \text{ and thus, } k > r.) \\ &= a_{i'} + rd + (k-r-1)d + a_{m-i'+1}, \text{ by (i)} \\ &= a_1 + a_m + (k-1)d, \text{ by (ii)} \\ &= a_1 + a_n, \text{ by (i).} \end{aligned}$$

[Necessity] Now we prove that (2) implies conditions (i) and (ii). As before, we view the sequence as consisting of blocks of m terms. By truncating the sequence to an odd number of blocks, we obtain a result for the 'centre' block and in particular, show that (ii) must hold. We establish another result by truncating the sequence to an even number of blocks and finally use Mathematical Induction to show that (i) holds as well.

Let $n = (2k-1)m$. Then by (2), for each positive integer k, $a_{(k-1)m+i} + a_{km-i+1} = a_{(k-1)m+1} + a_{km}$ holds for all i, $1 \le i \le m$. In particular, (ii) holds.

Now, let $a_{m+1} - a_1 = d$. We first consider $n = 2m$. For each i, $1 \le i \le m$, we have

$$\begin{aligned} a_1 + a_{2m} &= a_i + a_{2m-i+1}, \\ a_{2m} - a_{2m-i+1} &= a_i - a_1, \\ a_{2m} - (a_{2m} + a_{m+1} - a_{m+i}) &= a_i - a_1, \end{aligned}$$

$$a_{m+i} - a_i = a_{m+1} - a_1,$$
$$a_{m+i} - a_i = d.$$

Let $P(k)$ be the statement $a_{km+i} - a_{(k-1)m+i} = d$ for each i, $1 \le i \le m$, where k is a positive integer. The statement is true for $k = 1$. Assume the statement is true for all positive integers not exceeding k. We have for each i, $1 \le i \le m$,

$$a_{(k+1)m+i} + a_{km-i+1} = a_{km+i} + a_{(k+1)m-i+1},$$
(here we take $n = (2k-1)m$)
$$a_{(k+1)m+i} - a_{km+i} = a_{(k+1)m-i+1} - a_{km-i+1},$$
$$a_{(k+1)m+i} - a_{km+i} = a_{km+(m+1-i)} - a_{(k-1)m+(m+1-i)},$$
$$a_{(k+1)m+i} - a_{km+i} = d \qquad \text{(by induction hypothesis).}$$

By Mathematical Induction, $P(k)$ is true for all positive integers k. The property (i) follows. □

Observe that (2) is a more general condition than (1).

Example: The following sequences satisfy (2) but not (1):

(1) 2, 8, 5, 11, 8, 14, 11, 17, 14, 20, 17, 23, …
where $a_r = \begin{cases} 2+3(n-1) & if \quad r = 2n-1 \\ 8+3(n-1) & if \quad r = 2n \end{cases}$.
Note that here $m = 2$ and $d = 3$.

(2) 5, 1, 6, 2, 3, -1, 4, 0, 1, -3, 2, -2, -1, -5, 0, -4, …
Here $m = 4$ and $d = -2$.

Exercise: Find the sum to 50 terms of each of the sequences above.

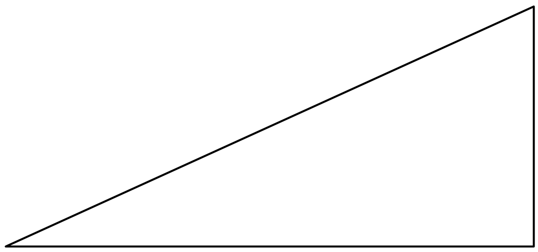
Now suppose the sequence $\{a_i\}_{i=1}^{\infty}$ satisfies condition (1) for every n. Take $m = 1$. Then, by condition (i) of Theorem 1, we obtain that $a_{i+1} = a_i + d$ holds for every i. So we have the following corollary.

Corollary 2: Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of real numbers. Then $\{a_i\}_{i=1}^{\infty}$ satisfies condition (1) only if it is an Arithmetic Progression.

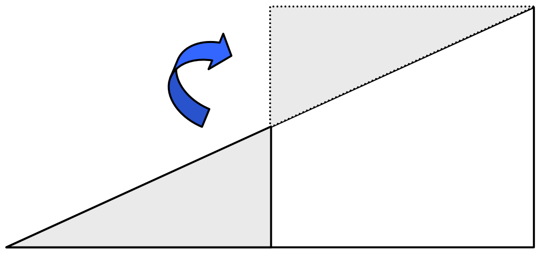
The results above essentially tell us that a sequence whose sum to n terms can be found using Gauss's method is either an AP or the combination of some APs.

The Continuous Case
We now move to the continuous case.

Consider the following right-angled triangle:



The area of the triangle above equals the area of the following rectangle which is obtained by cutting the triangle at the middle and pasting the two parts as shown:



This approach is similar to Gauss's method in obtaining the sum of a sequence. In fact, if we assume the equation of the diagonal is $f(x) = mx + d$, then for every $n \geq 0$ and $0 \leq x \leq n$, $f(x) + f(n{-}x) = f(0) + f(n)$. Once again this idea applies to more general cases.

By a simple computation, we can prove the following result:

Theorem 3: If $f : [0, n] \to \Re$ is a function such that $f(x) + f(n{-}x) = k$ holds for every $x \in [0, n]$, where k is a fixed number, then $\int_0^n f(x)dx = \frac{nk}{2}$.

Example:

(i) Since $\sin^2 x + \sin^2(\frac{\pi}{2} - x) = \sin^2 x + \cos^2 x = 1$, we have

$$\int_0^{\pi/2} \sin^2(x)dx = \frac{\pi/2\,(1)}{2} = \frac{\pi}{2}.$$

(ii) Let $f(x) = mx + c$. Since $f(x) + f(n-x) = f(0) + f(n) = c + mn + c = mn + 2c$, we have

$$\int_0^n f(x)dx = \frac{n(mn+2c)}{2}.$$

Note that when $f(x) = \sin^2(x)$, then $f(x) + f(n-x) = k$ only for $n = \frac{m\pi}{2}$, where m is an odd integer. Thus, of the two functions in the example above, only one of them, i.e., the linear function, satisfies $f(x) + f(n-x) = k$ for all values of n. We are thus led to pose the second problem.

Problem 2: Suppose $f: \Re_0^+ \to \Re$ is a function satisfying the following condition:

for every real number $n \geq 0$, $f(x) + f(n-x) = f(0) + f(n)$ holds for all x, $0 \leq x \leq n$. (3)

Must f be a linear function?

Let $\Re$ be the set of real numbers. Then $\Re$ is a vector space over the field Q of rational numbers, where the addition is the ordinary addition of numbers and the scalar multiplication is the ordinary multiplication of numbers. Then there exists a basis B for this vector space and we can assume that all elements in B are positive numbers. Thus every real number can be expressed as a linear combination of a finite number of elements of B (see, for example, Megginson (1998)).

In Theorem 4, we show that unlike in the discrete case where only the AP satisfies (1), there are infinitely many functions that satisfy (3).

Theorem 4: Let $f: \Re_0^+ \to \Re$ be a function and $B = \{v_i > 0 \mid i \in I\}$ be a basis for the vector space $\Re$ of all real numbers over the field Q of all rational numbers. Then the following two conditions are equivalent for f:

(1) $f(x) + f(n-x) = f(0) + f(n)$ holds for every $n \geq 0$ and $0 \leq x \leq n$;

(2) for each $x = \sum_{i \in I_x} a_i v_i$ ($a_i \in Q$), where I_x is a finite subset of I,

$$f(x) = f(0) + \sum_{i \in I_x} a_i (f(v_i) - f(0)) .$$

Proof (1) $\Rightarrow$ (2): To make the approach easier, we first assume that $f(0) = 0$. Then it is easy to see that for all $x, y \geq 0$, $f(x+y) = f(x) + f(y)$. We shall now obtain a result involving mainly rational numbers. This will serve as a stepping-stone for the more general real numbers. Let n and k be any two positive integers and x be any nonnegative number. We have

$$f(\tfrac{1}{n}x) + f(\tfrac{k-1}{n}x) = f(\tfrac{k}{n}x),$$
$$f(\tfrac{1}{n}x) + f(\tfrac{k-2}{n}x) = f(\tfrac{k-1}{n}x),$$
$$f(\tfrac{1}{n}x) + f(\tfrac{k-3}{n}x) = f(\tfrac{k-2}{n}x),$$
$$\vdots$$
$$\vdots$$
$$f(\tfrac{1}{n}x) + f(\tfrac{1}{n}x) = f(\tfrac{2}{n}x) .$$

Adding up the equations we have $kf(\tfrac{1}{n}x) = f(\tfrac{k}{n}x)$. If $k = n$, then $f(\tfrac{1}{n}x) = \tfrac{1}{n}f(x)$. Thus, $f(\tfrac{k}{n}x) = \tfrac{k}{n}f(x)$ and so we have that for any positive rational number r and any positive real number x, $f(rx) = rf(x)$. The last equation obviously holds for $r = 0$ and $x \geq 0$ because $f(0) = 0$.

Recall that each v_i is positive. Thus, in particular, we have $f(r_i v_i) = r_i f(v_i)$ for all ≥ 1. Now consider any real number $x = \sum_{i \in I_x} a_i v_i$ ($a_i \in Q$), where I_x is a finite subset of I. We may assume that $I_x = \{1, 2, \ldots, m\}$. We may further assume that $x = \sum_{i=1}^{m} a_i v_i$, where $a_i \geq 0$ if and only if $1 \leq i \leq l$ for some $l \leq m$. Thus, we have the following:

$$x = \sum_{i=1}^{l} a_i v_i + \sum_{i=l+1}^{m} a_i v_i ,$$

(We split the summation into two parts to separate terms according to the parity of a_i. This is in consideration of the fact that the domain of f is the set of non-negative real numbers.)

$$x - \sum_{i=l+1}^{m} a_i v_i = \sum_{i=1}^{l} a_i v_i ,$$

$$x + \sum_{i=l+1}^{m} b_i v_i = \sum_{i=1}^{l} a_i v_i , \text{ where } b_i = -a_i$$

$$f(x + \sum_{i=l+1}^{m} b_i v_i) = f(\sum_{i=1}^{l} a_i v_i),$$

$$f(x) + \sum_{i=l+1}^{m} f(b_i v_i) = \sum_{i=1}^{l} f(a_i v_i),$$

$$f(x) + \sum_{i=l+1}^{m} b_i f(v_i) = \sum_{i=1}^{l} a_i f(v_i),$$

$$f(x) = \sum_{i=1}^{l} a_i f(v_i) - \sum_{i=l+1}^{m} b_i f(v_i),$$

$$f(x) = \sum_{i=1}^{l} a_i f(v_i) + \sum_{i=l+1}^{m} a_i f(v_i),$$

$$f(x) = \sum_{i=1}^{m} a_i f(v_i).$$

Now if $f(0) \neq 0$, the result can be obtained by translation of the function by $f(0)$ as follows. Let $g(x) = f(x) - f(0)$. Thus, $f(x) + f(n-x) = f(n) + f(0)$ implies $g(x) + g(n-x) = g(n) + g(0)$. Since $g(0) = 0$, we have by the earlier argument, $g(x) = \sum_{i=1}^{m} a_i g(v_i)$.

Then we have:

$$\begin{aligned} f(x) &= g(x) + f(0) \\ &= \sum_{i=1}^{m} a_i g(v_i) + f(0) \\ &= f(0) + \sum_{i=1}^{m} a_i (f(v_i) - f(0)). \end{aligned}$$

(2) $\Rightarrow$ (1): Let n and x be any two positive numbers with $x \leq n$. We may assume that $x = \sum_{i=1}^{l} a_i v_i$ and $n = \sum_{i=1}^{l} b_i v_i$, where a_i, $b_i \geq 0$. Then we have:

$$\begin{aligned} f(x) + f(n-x) &= f(0) + \sum_{i=1}^{l} a_i (f(v_i) - f(0)) + f(\sum_{i=1}^{l} b_i v_i - \sum_{i=1}^{l} a_i v_i) \\ &= f(0) + \sum_{i=1}^{l} a_i (f(v_i) - f(0)) + f(\sum_{i=1}^{l} (b_i - a_i) v_i) \\ &= f(0) + \sum_{i=1}^{l} a_i (f(v_i) - f(0)) + f(0) + \sum_{i=1}^{l} (b_i - a_i)(f(v_i) - f(0)) \\ &= 2f(0) + \sum_{i=1}^{l} (b_i - a_i + a_i)(f(v_i) - f(0)) \\ &= 2f(0) + \sum_{i=1}^{l} b_i (f(v_i) - f(0)) \end{aligned}$$

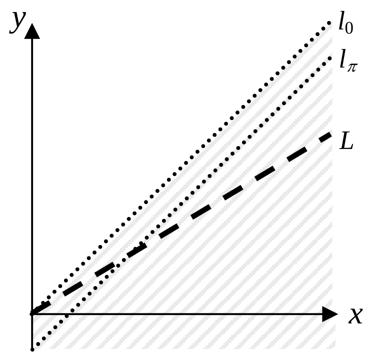
$= f(0) + f(n)$. □

Obviously, every function of the form $g(x) = mx + c$ (its graph is a straight line) satisfies (3). The following is an example of a function satisfying (3) whose graph is not a straight line.

Example: Let $B = \{v_i \mid i \in I\}$ be a basis for the vector space $\Re$ of real numbers over the field Q of rational numbers with $v_i > 0$, $v_1 = 1$ and $v_2 = \sqrt{2}$. Define $f : \Re_0^+ \to \Re$ as follows:

$f(x) = \sum_{i=1}^{m} a_i f(v_i)$ and $f(v_i) = \begin{cases} 2\sqrt{2} & if \quad i = 2 \\ v_i & otherwise \end{cases}$, where $x = \sum_{i=1}^{m} a_i v_i$ for some positive integer m and a_i's from Q.

Let $A = \{ \sum_{i=1}^{m} a_i v_i \mid a_2 = 0\}$. For each $c \in A$, let $A_c = \{ c + r\sqrt{2} \in \Re^+ \mid r \in Q - \{0\}\}$. We observe that $\Re_0^+ = A \cup \bigcup_{c \in A} A_c$ and $A_c \cap A_{c'} = \phi$ for $c \neq c'$. The graph of the restriction of $f(x)$ over A is the dotted line $L = \{(x, x) \mid x \in A\}$. The graph of the restriction of $f(x)$ over A_c is the dotted line $l_c = \{(x, 2x - c) \mid x \in A_c\}$. The figure below shows a sketch graph of $y = f(x)$. The line L and two of the lines l_c, i.e., l_0 and l_π, are shown. The graph of $y = f(x)$ is the shaded region which can be visualized as the union of the line L and lines l_c.



Some guidelines for use in classroom

The teacher, now familiar with the results of sections 1 and 2, may want to consider the following guidelines on how these results can be explored and discovered in the classroom.

1. Begin with the story of young Gauss, his method and lead to condition (1). Help students to identify sequences which satisfy (1). Then ask what mathematical question (that is, identify ALL sequences) naturally follows. Discuss different ways of asking the same question – this will probably lead on to the converse question of Problem 1. At this point, a necessary digression may be in the form of explaining the logical statement and its converse, inverse and contrapositive. Teachers may wish to refer to mathematical dictionaries such as Nichols and Schwartz (1995) for these and other definitions.

2. Have the students discuss, either in class or as homework, approaches to solving Problem 1. Hints may be given and in our particular approach, the heuristic of 'solving the particular by using the general' may be suggested. If Problem 1 were solved as is, the teacher may encourage the students to seek a generalization of the result to obtain Theorem 1.

3. Introduce the discrete-continuous dichotomy seen very often in mathematics, such as in summation and integration, and discrete and continuous random variables. Lead on to the continuous version of the problem, Problem 2. At this point, more help may be needed and teachers can give the essentials of linear algebra as a lecture or a reading task. The proof of Theorem 4 is rather long and it is unlikely that many students except the best will be able to come up with it on their own. Instead, encourage exploration to obtain different functions that fulfill condition (3) and seek to generalise them. The final proof can be in a guided form such as the Cloze passage suggested by Tay (2001).

Conclusion

In this paper we explored the possible applications of an old but very useful method to a larger class of sequences. This exploration serves as a good example of how one can investigate a method or a result in mathematics by first posing a problem based on the existing method or theorem. Here we applied the converse problem posing method, one of the most popularly used problem posing strategies. The immediate benefit to the reader is a deeper understanding of Gauss's method and its limitations. Using the result obtained here, teachers can produce problems of more variety related to Gauss's methods for their students or adopt the approach here to propose mathematical projects. Students may follow suit to explore other learned methods or theorems, thus make their learning more active and interesting.

Finally, the reader may wish to explore along the same lines with the Geometric Progression. One may begin by trying the following problem:

Problem 3: Suppose $\{a_i\}_{i=1}^{\infty}$ is a sequence such that $S_n = \dfrac{a_n - a_1}{r-1}$ for every *n*, where S_n is the sum to *n* terms and *r* is some constant real number. Must $\{a_i\}_{i=1}^{\infty}$ be a Geometric Progression?

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